GENERALIZED VIRTUAL DISPLACEMENTS

(OBOBSHCHENNYE VIRTUAL'NYE PEREMESHCHENIIA)

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The author proposes a generalization of the definition of virtual displacements, which could be applied to ideal constraints and also to certain other interactions of the points of a system and constraints. The virtual displacements would depend on the law of interaction; and for a certain special form of this law we have the commonly accepted definition of virtual displacements.

1. Let us consider a system of N material points, whose positions at every instant of time t are determined by $s = 3N$ generalized coordinates $q_1, \ldots, q_s$, subject to l constraints

$$f_i\left(t,\ q_k,\ \frac{dq_k}{dt}, \ldots,\ \frac{d^j q_k}{dt^j}\right) = 0 \qquad (i = 1, \ldots, l) \tag{1.1}$$

We assume here that the constraints may limit not only the positions of the points and their velocities, but the accelerations as well. The functions f_i are assumed to be differentiable with respect to all their arguments. The formula (1.1) indicates concisely that each of f_i's could depend on all coordinates $q_1, \ldots, q_s$, and on their derivatives up to the order j (it is of course possible that some of the f_i's would depend on the derivatives up to the order p_i, $0 < p_i < j$). The integral or integro-differential constraints will not be considered.

Independently of the character of the constraints and the nature of the physical interaction of the points of the system, the equations of motion may be written in the form

$$R_k + Q_k + \frac{\partial T}{\partial q_k} - \frac{d}{dt}\frac{\partial T}{\partial \dot q_k} = 0 \qquad \left(\dot q_k = \frac{dq_k}{dt}\right) \tag{1.2}$$

where Q_k, R_k and T are the generalized forces, reactions and the kinetic energy, respectively, with $k = 1, 2, \ldots, s$.

Consider the system of functions

in which the arbitrary differentiable functions $\eta_k(t)$ have the same dimension as the corresponding generalized coordinate q_k; the infinitely small parameter α does not depend on t and could have a dimension. Multiplying the k-th equation (1.2) by $\delta\psi_k$ and adding, we obtain

$$\sum_{k=1}^{s}\left(R_k+Q_k+\frac{\partial T}{\partial q_k}-\frac{d}{dt}\frac{\partial T}{\partial \dot q_k}\right)\delta\psi_k=0 \qquad (1.4)$$

It is clearly evident that if the relationship (1.4) is valid for an arbitrary $\delta\psi_k$, then any system of equations (1.2) must also be valid. Equation (1.4) could serve as a general equation of mechanics for the most general kind of motion with completely arbitrary constraints. It is essential here that the quantities $\delta\psi_k$ be completely arbitrary. They do not have to be variations of the generalized coordinates in the usual sense, and their dimensions do not have to be the same as the dimensions of the corresponding coordinates.

Equations (1.4) contain the initially unknown reactions of constraints R_k, which have to be eliminated when equations (1.1) and the law of interaction of the points of the system with the constraints are to be used. It is well known how this is done in the case of ideal constraints. In the general case, the constraints are arbitrary, and there exists a set of quantities $\delta\psi_k$ for which the relation

$$\sum_{k=1}^{s}R_k\delta\psi_k=\sum_{k=1}^{s}\left[\varphi_{0k}\delta\psi_k+\sum_{m=1}^{\beta}\sum_{p=1}^{m}\frac{d}{dt}\left(\varphi_{mk}{}^{p}\frac{d^{m-p}}{dt^{m-p}}\delta\psi_k\right)\right] \qquad (1.5)$$

is satisfied at every instant of time. Here ϕ_{ok} are given functions of the time, of the coordinates, and of their time derivatives; these functions could be the resistance forces of the medium, or some supplementary active forces; the functions $\phi_{mk}{}^{p}$ will be explained later. The condition, which in the most general case of physical interactions determines the set $\delta\psi_k$, may be written as

$$\Phi_i\left(t,\ q_k,\ \frac{dq_k}{dt},\ldots,\ \frac{d^{\nu}q_k}{dt^{\nu}},\ f_r,\ldots,\ \delta\psi_k,\ \frac{d}{dt}\delta\psi_k,\ldots,\ \frac{d^{\gamma}}{dt^{\gamma}}\delta\psi_k\right)=0 \qquad (1.6)$$

$$(i=1,\ldots,l)$$

where Φ_i are certain functions differentiable with respect to all their arguments. These functions may not be completely arbitrary. Indeed, the quantities $\delta\psi_k$ should be determinable with the accuracy of the multiplier α. Hence, Φ_i should be homogeneous functions of some order n_i with respect to the whole set of the quantities $\delta\psi_k$ and their derivatives. In the simplest case $n_i = 1$ $(i = 1, \ldots, l)$, Φ_i would be homogeneous linear functions, and the relation (1.6) could be written as

$$\delta\psi_k=\alpha\eta_k(t) \qquad (1.3)$$

$$\sum_{k=1}^{s} \sum_{m=0}^{\gamma} a_{ik}{}^{m} \frac{d^m}{dt^m} \delta\psi_k = 0 \qquad (i = 1, \ldots, l) \tag{1.7}$$

Besides, the r-th equation of the constraints (1.1) is determined with the accuracy of the constant multiplier C_r, therefore Φ_i and also $a_{ik}{}^{m}$ should be homogeneous functions of some order n_{ir} with respect to each variable of the group f_r, $\partial f_r/\partial q_k$, ...

We shall call the quantities $\delta\psi_k$, which satisfy equations (1.6), or in special cases equations (1.7) the general virtual displacements, or simply the $\delta\psi$-displacements.

Generally speaking, the functions Φ_i, $a_{ik}{}^{m}$ depend on all functions $f_1, \ldots, f_l$ and on all their partial derivatives up to a certain order, but in special cases each of the Φ_i could depend only on one of the functions f_i and on its derivatives. Different choices of Φ_i, $a_{ik}{}^{m}$ are mathematical expressions of different characters of the physical interaction. For a definite law of the physical interaction, the functions Φ_i and $a_{ik}{}^{m}$ assume a definite form, and conversely, choosing a certain form for Φ_i, and $a_{ik}{}^{m}$ we select at the same time some definite law for the physical interaction. The general case (1.6) is quite difficult, therefore from now on we shall study the virtual displacements (1.7), which, of course, limits the class of constraints to be considered. Nevertheless, this narrower class is sufficiently comprehensive, containing the ideal constraints and also the holonomic constraints with friction.

2. Usually, the virtual displacements are determined from systems of algebraic equations, but the $\delta\psi$-displacements are determined from (1.7), which are not algebraic but differential equations. The $\delta\psi_k$ functions determined from these equations are in general completely different from the commonly accepted virtual displacements.

We shall derive now the equations of motion of a system with constraints (1.1) and conditions (1.7).

The general equation (1.4) with the conditions (1.5) and $\beta > 0$ will assume the form

$$\sum_{k=1}^{s} \left[\left(\varphi_{0k} + Q_k + \frac{\partial T}{\partial q_k} - \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} \right) \delta\psi_k + \sum_{m=1}^{\beta} \sum_{p=1}^{m} \frac{d}{dt} \left(\varphi_{mk}{}^{p} \frac{d^{m-p}}{dt^{m-p}} \delta\psi_k \right) \right] = 0 \tag{2.1}$$

(when $\beta = 0$, the summation sign with respect to m is absent). Multiplying the ith equation (1.7) by an undetermined multiplier λ_i, summing with respect to i and combining with (2.1), we obtain

$$\sum_{k=1}^{s}\left[\left(\varphi_{0k}+Q_k+\frac{\partial T}{\partial q_k}-\frac{d}{dt}\frac{\partial T}{\partial \dot q_k}\right)\delta\psi_k+\right.$$
$$\left.+\sum_{m=1}^{\beta}\sum_{p=1}^{m}\frac{d}{dt}\left(\varphi_{mk}{}^{p}\frac{d^{m-p}}{dt^{m-p}}\delta\psi_k\right)+\sum_{i=1}^{l}\sum_{m=0}^{\gamma}\lambda_i a_{ik}{}^{m}\frac{d^m}{dt^m}\delta\psi_k\right]=0 \qquad (2.2)$$

Using the identity

$$\lambda_i a_{ik}^m\frac{d^m}{dt^m}\delta\psi_k=(-1)^m\left(\frac{d^m}{dt^m}\lambda_i a_{ik}{}^m\right)\delta\psi_k+$$
$$+\sum_{p=1}^{m}(-1)^{p+1}\frac{d}{dt}\left[\left(\frac{d^{p-1}}{dt^{p-1}}\lambda_i a_{ik}{}^m\right)\frac{d^{m-p}}{dt^{m-p}}\delta\psi_k\right]\quad (m>0) \qquad (2.3)$$

which can be verified by performing the indicated operations, we shall transform the quantity $\lambda_i a_{ik}{}^m (d^m/dt^m)\delta\psi_k$. The functions $\phi_{mk}{}^p$ and the quantity β will be expressed now as

$$\beta=\gamma,\ \varphi_{mk}{}^{p}=\sum_{i=1}^{l}(-1)^p\frac{d^{p-1}}{dt^{p-1}}(\lambda_i a_{ik}{}^m)\quad (m>0) \qquad (2.4)$$

Substituting (2.3), (2.4) in (2.2) and combining similar terms, we obtain

$$\sum_{k=1}^{s}\left[\varphi_{0k}+Q_k+\frac{\partial T}{\partial q_k}-\frac{d}{dt}\frac{\partial T}{\partial \dot q_k}+\sum_{i=1}^{l}\sum_{m=0}^{\gamma}(-1)^m\frac{d^m}{dt^m}(\lambda_i a_{ik}{}^m)\right]\delta\psi_k=0$$

from which, after the usual reasoning, we obtain the system of equations

$$\varphi_{0k}+Q_k+\frac{\partial T}{\partial q_k}-\frac{d}{dt}\frac{\partial T}{\partial \dot q_k}+\sum_{i=1}^{l}\sum_{m=0}^{\gamma}(-1)^m\frac{d^m}{dt^m}(\lambda_i a_{ik}{}^m)=0 \qquad (2.5)$$

which, together with (1.1), constitutes the full system of $(s + l)$ equations for $(s + l)$ functions q_k, λ_i.

Comparing (1.4) with (2.5), we can show that

$$R_k=\varphi_{0k}+\sum_{i=1}^{l}\sum_{m=0}^{\gamma}(-1)^m\frac{d^m}{dt^m}(\lambda_i a_{ik}{}^m) \qquad (2.6)$$

which means that actually the reactions of constraints depend not only on the analytic form of the constraints, but also on the form of the functions $a_{ik}{}^m$. The constraints with reactions (2.6) are of course not ideal. Besides, the conventional name "constraints with friction" becomes a misnomer here, because in the presence of conditions (1.7) there are possible also reactions, for which the total mechanical energy of the system increases, although the active forces are performing negative work.

In order to verify equations (2.5), it is necessary to check whether conditions (1.5), which were essential in the derivations of these equations, are satisfied. To perform this, we shall substitute the values of R_k from (2.6) in (1.5), replace β by γ, and replace ϕ_{mk}^p by their values obtained from (2.4). As a result we shall have for $\gamma > 0$

$$\sum_{k=1}^{s}\sum_{i=1}^{l}\left\{\sum_{m=0}^{\gamma}(-1)^m\left(\frac{d^m}{dt^m}\lambda_i a_{ik}{}^m\right)\delta\psi_k -\right.$$
$$\left.-\sum_{m=1}^{\gamma}\sum_{p=1}^{m}(-1)^p\frac{d}{dt}\left[\left(\frac{d^{p-1}}{dt^{p-1}}\lambda_i a_{ik}{}^m\right)\frac{d^{m-p}}{dt^{m-p}}\delta\psi_k\right]\right\} = 0$$

(for $\gamma = 0$ the second sum would be absent); hence, taking into account (2.3), we obtain the relation

$$\sum_{i=1}^{l}\sum_{k=1}^{s}\sum_{m=0}^{\gamma}\lambda_i a_{ik}{}^m\frac{d^m}{dt^m}\delta\psi_k = 0$$

which must be satisfied on the strength of (1.7). The verification of equations (2.5) is completed.

In general, equations (2.5) are of higher order than two. To determine now the arbitrary constants of integration we must know not only the initial generalized coordinates and velocities, but also their derivatives of higher order. All these quantities must, of course, satisfy the equations of constraints (1.1) and also the relations obtained from successive differentiation of (1.1) with respect to time, since derivatives of higher order than j are needed for the determination of the constants. The number of arbitrary constants depends on the form of the functions $a_{ik}{}^m$.

As our constraints now are not ideal, we need not, in general, select new generalized coordinates in order to eliminate the holonomic constraints. Nevertheless, in certain cases, we could also obtain equations analogous to the Lagrange equations of the second kind. Let us assume that all the constraints are holonomic and $\gamma = 0$; we shall select new generalized coordinates such that the constraints become identities. From equations (1.7) which become in this case

$$\sum_{k=1}^{s} a_{ik}{}^\circ\delta\psi_k = 0 \qquad (i = 1, \ldots, l) \tag{2.7}$$

we shall express any quantities $\delta\psi_k$ in terms of the remaining ones. We shall substitute the obtained expressions for the dependent $\delta\psi_k$ in equation (2.1), putting $\beta = \gamma = 0$, that is, in the equation

$$\sum_{k=1}^{s}\left(\varphi_{0k} + Q_k + \frac{\partial T}{\partial q_k} - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k}\right)\delta\psi_k = 0$$

We shall express coefficients of the independent $\delta\psi_k$ as functions of the above mentioned new generalized coordinates and set them equal to zero. The equations thus obtained would contain a minimum number of unknown functions. Besides, when $\gamma = 0$, a similar transformation could be carried out for nonholonomic constraints of the form

$$\dot{q}_i = \varphi_i(t, q_{l+1}, \ldots, q_s, \dot{q}_{l+1}, \ldots, \dot{q}_s) \qquad (i = 1, \ldots, l) \tag{2.8}$$

if $\phi_{ok} + Q_k$, T do not depend on $q_1, q_2, \ldots, q_s$ (but could depend on their derivatives).

3. We shall study some actual examples of $\delta\psi$-displacements. Let the functions $a_{ik}{}^m$ be defined by the formulas

$$a_{ik}{}^m = \frac{\partial f_i}{\partial b_{mk}} \qquad b_{mk} = \frac{d^m q_k}{dt^m} \qquad (m = 0, 1, \ldots, \gamma;\ i = 1, \ldots, l) \tag{3.1}$$

Equations (1.7), which determine the $\delta\psi$-displacements, then become

$$\delta f_i \equiv \sum_{k=1}^{s} \sum_{m=0}^{j} \frac{\partial f_i}{\partial b_{mk}} \frac{d^m}{dt^m} \delta q_k = 0 \qquad (i = 1, \ldots, l) \tag{3.2}$$

where the symbol δf_i denotes an isochronous variation of f_i, that is

$$\delta f_i = \alpha \left\{ \frac{d}{d\alpha} f_i \left[t, q_k + \delta q_k, \frac{d}{dt}(q_k + \delta q_k), \ldots, \frac{d^j}{dt^j}(q_k + \delta q_k) \right] \right\}_{\alpha=0} \tag{3.3}$$

The quantities $\delta\psi_k$ are determined by formulas (1.3), and in this case it is convenient to replace them by q_k. When $j = 0$, we obtain from (3.2) the conventional definition of virtual displacements for holonomic constraints; for nonholonomic constraints ($j = 1$) the definition (3.2) differs from the conventional one. It must be mentioned though, that from the mathematical point of view, definition (3.2) is much more natural than the conventional one. It seems that constraints with a similar rule of interaction were not investigated previously, because it was not known how to construct equations (2.5). From the general point of view definition (3.2) and the conventional one are equally valid; they correspond in the case $j = 1$ to a different nature of interaction of the points in the system and of the constraints.

Equations (2.5) become here

$$\varphi_{0k} + Q_k + \frac{\partial T}{\partial q_k} - \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} + \sum_{i=1}^{l} \sum_{m=0}^{j} (-1)^m \frac{d^m}{dt^m} \left(\lambda_i \frac{\partial f_i}{\partial b_{mk}} \right) = 0 \tag{3.4}$$

The above equations, together with (1.1), form the complete system of $(s + l)$ equations. We shall estimate now the number of independent arbitrary constants, assuming that equations (3.4) are of $2j$th order with respect to all the coordinates $q_k (j > 1)$. Let us assume that among the constraints (1.1) there are r_0 holonomic constraints, r_1 constraints which enforce limitations on positions and velocities, that is, constraints of the first order, r_2 constraints of the second order, and finally r_j constraints of the jth order. We assume that $r_j \neq 0$. Differentiating the constraints of the mth order $(2j - m)$ times with respect to time, we obtain, together with (3.4), a system of $(s + l)$ equations of jth order with respect to the coordinates q_k and of mth order with respect to λ_i corresponding to the constraints of the mth order. The solution of the resulting system will depend on

$$2js + \sum_{m=1}^{j} mr_m$$

arbitrary constants, not all of them independent. Indeed, the resulting solution must satisfy all the constraints and all the relations obtained from the successive differentiation of the constraint equations with respect to time. Hence, the constraints of the mth order will give in addition $r_m(2j - m)$ independent relations. Thus, the number of independent arbitrary constants in our case would equal

$$2\left[js - \sum_{m=0}^{j} (j - m) r_m\right] \tag{3.5}$$

If the order of the resulting equations with respect to any of the coordinates is lower (higher) than $2j$, then the number of the independent constants is correspondingly lower (higher).

The arbitrary constants (3.5) could be determined from the requirement that the system at two prescribed instants of time occupy a certain prescirbed position with prescribed velocities, accelerations, and so on, up to the $(j - l)$th derivative inclusive, $(j \geqslant 1)$. In particular, for holonomic constraints $(j = 0)$ and nonholonomic $(j = 1)$ we could prescribe two arbitrary positions of the system at two prescribed instants of time, which are consistent with the holonomic constraints. It should be mentioned that for the constraints of Chetaev [1], that is, constraints for which (1.7) becomes $(j = 1)$,

$$\sum_{k=1}^{s} \frac{\partial f_i}{\partial \dot{q}_k} \delta\psi_k = 0 \qquad (i = 1, \ldots, l) \tag{3.6}$$

this could not be done. For the constraints (3.6), and with the same conditions otherwise, we obtain a solution which depends only on $2s - 2r_0 - r_1$ constants. When $r_1 > 0$, then the number of disposable constants is less than $2(s - r_0)$, consequently with the constraints (3.6) we could not prescribe arbitrarily two positions of the same system, if we take into account only the holonomic constraints. As far as we know this fact, so fundamental for the formulation of the Hamilton-Ostrogradski principle, has not been previously noticed.

We shall consider now the case of cyclic coordinates for nonholonomic constraints of the first order $(j = 1)$. Let us mention that the holonomic constraints with conditions (3.2) become conventional ideal constraints; consequently, they could be eliminated by a suitable selection of generalized coordinates. Let us assume that the holonomic constraints are already eliminated and the nonholonomic constraints are solved with respect to the generalized velocities $\dot{q}_i$, that is

$$f_i \equiv \dot{q}_i - \varphi_i(t, q_{l+1}, \ldots, q_s, \dot{q}_{l+1}, \ldots, \dot{q}_s) = 0 \qquad (i = 1, \ldots, l) \qquad (3.7)$$

Besides, let $\phi_{oi} + Q_k = O_i (i = 1, \ldots, l)$, and let us assume that the remaining quantities $\phi_{ok} + Q_k$ and T do not depend on the cyclic coordinates $q_1, \ldots, q_l$. Then equations (3.4) would yield the system

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} + \frac{d\lambda_i}{dt} = 0 \qquad (i = 1, \ldots, l) \qquad (3.8)$$

$$\varphi_{0k} + Q_k + \frac{\partial T}{\partial q_k} - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \sum_{i=1}^{l}\left[\lambda_i \frac{\partial \varphi_i}{\partial q_k} - \frac{d}{dt}\left(\lambda_i \frac{\partial \varphi_i}{\partial \dot{q}_k}\right)\right] = 0 \qquad (k = l+1, \ldots, s)$$

Solving the first group of equations (3.8) we find

$$\lambda_i = c_i - \frac{\partial T}{\partial \dot{q}_i} \qquad (i = 1, \ldots, l) \qquad (3.9)$$

where c_i are arbitrary constants.

Utilizing the function $T_0 = T(t, q_{l+1}, \ldots, q_s, \dot{\phi}_1, \ldots, \dot{\phi}_l, \dot{q}_{l+1}, \ldots, \dot{q}_s)$ from the second group of equations (3.8), we obtain the system

$$\varphi_{0k} + Q_k + \frac{\partial T_0}{\partial q_k} - \frac{d}{dt}\frac{\partial T_0}{\partial \dot{q}_k} = \sum_{i=1}^{l} c_i\left(\frac{\partial \varphi_i}{\partial q_k} - \frac{d}{dt}\frac{\partial \varphi_i}{\partial \dot{q}_k}\right) \qquad (k = l+1, \ldots, s) \quad (3.10)$$

analogous to the equations of Chaplygin. It should be mentioned that for $c_i = 0$ $(i = 1, \ldots, l)$ equations (3.10) coincide with the erroneously derived equations of Lindeloff [2]. The Lindeloff equations refer to the motion of a rigid body on a plane without sliding, but his law of interaction differs from that of Chaplygin and from the one Lindeloff himself had assumed.

4. We shall verify the validity of the Hamilton-Ostrogradski principle for the motion of a system with constraints (1.1) and conditions (3.2). For the sake of simplicity, we shall limit ourselves to a conservative system. We recall that we may prescribe m derivatives $(m \geqslant 0)$ of the generalized coordinates up to the $(j-1)$th order, which satisfy all constraints up to $(j-1)$th order inclusive, for two instants of the time, t_1 and t_2. This means that at these two instants of time we could set equal to zero all the $d^n\delta q_k/dt^n (n = 0, 1, \ldots, j-1)$. In the case of the constraints (3.6) we could also set equal to zero all δq_k, that is, to prescribe positions for the system at two instants of time; in this last

case, however, the existence of the solution for every boundary condition is not guaranteed.

The Hamilton-Ostrogradski principle could be formulated as follows: the motion of a system actually taking place differs from all other possible motions consistent with the constraints (1.1) and with the same initial and final positions by the fact that the functional

$$J = \int_{t_1}^{t_2} \Big(L + \sum_{i=1}^{l} \lambda_i f_i\Big) dt \tag{4.1}$$

assumes a stationary value. Here L is the Lagrangian, λ_i are undetermined multipliers. To prove the principle it is sufficient to compare the Euler equations for the functional J with equations (3.4) and take into account also that

$$\Big(\frac{d^m}{dt^m}\,\delta q_k\Big)_{t=t_1} = \Big(\frac{d^m}{dt^m}\,\delta q_k\Big)_{t=t_2} = 0 \qquad (m = 0, 1, \ldots, j-1) \tag{4.2}$$

Let us consider the interesting case of canonical variables, introducing first new independent variables b_{mk} $(m = 0, 1, \ldots, j)$ (see (3.1)) in the functional J, for which conditions (4.2) assume the form

$$(\delta b_{mk})_{t=t_1} = (\delta b_{mk})_{t=t_2} = 0 \qquad (m = 0, 1, \ldots, j-1) \qquad \Big(\delta b_{mk} = \alpha \frac{d^m}{dt^m}\,\eta_k(t)\Big) \tag{4.3}$$

In what follows we shall utilize the following obvious relation

$$\delta b_{mk} = \frac{d^m}{dt^m}\,\delta q_k \tag{4.4}$$

The introduction of the additional conditions

$$b_{mk} - \frac{d}{dt}\, b_{(m-1)k} = 0 \qquad (m = 1, 2, \ldots, j) \tag{4.5}$$

will not change the extremum of the functional [3].

Following the customary procedure, we shall seek the extremum of the functional $J(b_{ok}, b_{1k}, \ldots, b_{mk})$ with conditions (4.5) and also seek the absolute extremum of the functional

$$J_1 = \int_{t_1}^{t_2} \Big[L + \sum_{i=1}^{l} \lambda_i f_i + \sum_{m=1}^{j} \sum_{k=1}^{s} p_{mk} \Big(\frac{d}{dt}\, b_{(m-1)k} - b_{mk}\Big)\Big] dt \tag{4.6}$$

where p_{mk} are the undetermined Langrange multipliers.

It is easy to determine them in the case $j = 1$ from the Euler equations for J_1:

$$p_{1k} = \frac{\partial L}{\partial b_{1k}} + \sum_{i=1}^{l} \lambda_i \frac{\partial f_i}{\partial b_{1k}} \tag{4.7}$$

From relations (4.7) and from equations (1.1) we shall write down the functions

$$\begin{aligned} b_{1r} &= b_{1r}\,(t, q_k, p_{1k}, \ldots, p_{jk}) && (r = 1, \ldots, s) \\ \lambda_i &= \lambda_i\,(t, q_k, p_{ik}, \ldots, p_{jk}) && (i = 1, \ldots, l) \end{aligned} \tag{4.8}$$

Substituting the above values into (4.3) and denoting the quantity

$$\sum_{k=1}^{s} p_{1k} b_{1k} - L - \sum_{i=1}^{l} \lambda_i f_i = H\,(t, q_k, p_{1k}) \tag{4.9}$$

we shall rewrite the functional J_1 in the form

$$J_{11} = \int_{t_1}^{t_2} \Big(\sum_{k=1}^{s} p_{1k} \frac{dq_k}{dt} - H\Big)\, dt \tag{4.10}$$

For the functional J_{11} the Euler equations are the canonical Hamiltonian equations

$$\frac{dp_{1k}}{dt} = -\frac{\partial H}{\partial q_k}, \qquad \frac{dq_k}{dt} = \frac{\partial H}{\partial p_k} \tag{4.11}$$

We could apply to equations (4.11) the well-known methods of integration without any essential changes.

5. Example. We shall consider the motion of a material point of weight P with the following constraint on the velocity

$$\dot{q}_1^2 + \dot{q}_2^2 = a^2 = \mathrm{const} \tag{5.1}$$

(q_1 and q_2 are Cartesian coordinates, the q_2-axis is parallel to the gravity force).

Solving the constraint equation with respect to the derivative of the cyclic coordinate q_1 we obtain

$$q_1 = \pm\sqrt{a^2 - \dot{q}_2^2} \tag{5.2}$$

Utilizing equation (3.8), putting $\phi_{ok} = 0$, and noting that $T_0 = Pa^2/2g$, where g is the acceleration due to gravity, we obtain

$$P = \pm c_1 \frac{d}{dt} \frac{\dot{q}_2}{\sqrt{a^2 - \dot{q}_2^2}} \qquad \text{or} \qquad q_2 = c_3 \pm \frac{a}{P}\sqrt{c_1^2 + (Pt + c_2)^2} \tag{5.3}$$

Here c_1, c_2, c_3 are arbitrary constants. Utilizing the constraint equation and integrating the expression for q_1 we obtain

$$q_1 = c_4 \pm \frac{ac_1}{P} \ln [(Pt + c_2) + \sqrt{c_1^2 + (Pt + c_2)^2}] \tag{5.4}$$

In order to obtain the reaction we shall use equations (3.6), noting that in our case

$$R_1 = -\frac{d\lambda}{dt}, \qquad R_2 = -\frac{d}{dt}\left(\lambda \frac{\dot{q}_2}{\sqrt{a^2 - \dot{q}_2^2}}\right), \qquad \lambda = c_1 + \frac{P}{g}\dot{q}_1$$

The final results are

$$R_1 = \pm \frac{P^2 c_1 \dot{q}_2}{g[c_1^2 + (Pt + c_2)^2]}, \qquad R_2 = -P \mp \frac{P^2 c_1 \dot{q}_1}{g[c_1^2 + (Pt + c_2)^2]} \tag{5.5}$$

The resulting motion could be interpreted as follows: suppose there is a rocket with two reaction motors. One of them supplies a constant force component $R' = P$, which balances the force of gravity. The second motor supplies the reactive force

$$R'' = \frac{P^2 a c_1}{g[c_1^2 + (Pt + c_2)^2]}$$

along the normal to the trajectory and directed toward its concave side. The variation of the mass of the rocket is neglected. If we regard the constraint (5.1) as the Chetaev constraint, which means that we assume that the constraint imposes the following limitations

$$\dot{q}_1 \delta q_1 + \dot{q}_2 \delta q_2 = 0 \tag{5.6}$$

then it is easily seen that the reaction of this constraint is always tangent to the trajectory. It should be mentioned, however, that the solution of this example with conditions (5.6) depends not on four, but only on three arbitrary constants, consequently the trajectory could not connect two arbitrarily prescribed points.

6. Concluding, we shall present a simpler case for the determination of a_{ik}^* from the prescribed law of interaction. Suppose that a material point moves in a plane with Cartesian coordinates q_1 and q_2, with the frictional constraint

$$\varphi(q_1, q_2) = 0 \tag{6.1}$$

and the angle of friction τ is a prescribed function of the coordinates, velocities and their higher derivatives. We shall assume that the reaction is directed along grad ϕ. In order to determine the quantities a_{11}^0, a_{12}^0, we can construct the equations ($\gamma = 0$):

$$a_{11}^\circ \frac{\partial \varphi}{\partial q_1} + a_{12}^\circ \frac{\partial \varphi}{\partial q_2} = a\,|\operatorname{grad}\varphi|\cos\tau, \quad a_{11}^\circ \dot{q}_1 + a_{12}^\circ \dot{q}_2 = -a\dot{q}\sin\tau \tag{6.2}$$

where

$$a = \sqrt{(a_{11}^\circ)^2 + (a_{12}^\circ)^2}, \qquad \dot{q} = \sqrt{\dot{q}_1^2 + \dot{q}_2^2}$$

A solution of equations (6.2) could be obtained with an accuracy within an arbitrary coefficient, for example, in the form

$$a_{11}^\circ = \dot{q}_2 \,|\operatorname{grad}\varphi|\cos\tau + \dot{q}\frac{\partial\varphi}{\partial q_2}\sin\tau, \quad a_{12}^\circ = -\dot{q}_1\,|\operatorname{grad}\varphi|\cos\tau - \dot{q}\frac{\partial\varphi}{\partial q_1}\sin\tau \qquad (6.3)$$

The quantities a_{ik}^* could be determined similarly for a three-dimensional space.

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